

Robustness of multiparty nonlocality to local decoherence

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We investigate the robustness of multiparty nonlocality under local decoherence, acting independently and equally on each subsystems. To be specific, we consider an N -qubit GHZ state under depolarization, dephasing, or dissipation channel, and tested the nonlocality by violation of Mermin-Klyshko inequality, which is one of Bell's inequalities for multi-qubit systems. The results show that the robustness of nonlocality increases with the number of qubits, and that the nonlocality of an N -qubit GHZ state with even N is extremely persistent against dephasing.

I. INTRODUCTION

Nonlocality is one of the most surprising and important features of quantum physics. It is firstly referred by Einstein, Podolsky, and Rosen, and has been receiving an enormous attention and interest of scientists since Bell[1, 2] designed an inequality to expose it. Now, it is an essential part of quantum information science and quantum physics foundation.

However, nonlocality, like other quantum features, is easily destroyed by decoherence, and the concrete pictures of its destruction is not well known. One might gain insight into it by analyzing nonlocal states under decoherence. Werner[3] studied a maximally entangled two qubits under white noise, and Kazlikowski et. al.[4] extended Werner's work to a two d -dimensional system.

In particular, we are interested in the robustness of nonlocality according to the number of consisting particles. Nonlocality is hardly seen at macroscopic level because of strong decoherence, caused by many interactions with environment. In a previous study[5], it was shown that entanglement of an N -qubit GHZ state is more robust to depolarization as the number of qubits increases. The authors examined persistency of inseparability, which is the mathematical definition of entanglement. In this paper, we are interested in persistency of nonlocality, which is physically meaningful nature of entanglement.

We examine the decoherence process under realistic noise. Besides depolarization we consider two other common decoherence processes, dissipation and dephasing, which are called T_1 and T_2 process, respectively. For simplicity, we assume equal and independent local decoherence on each sub-particles. Cabello and Feito[6] studied Bell's inequality with depolarization and dephasing, and found that violations of Bell's inequality for a

two qubit system are extremely robust to dephasing. Although they extended their work to three and four qubits, they failed to show this extreme robustness, because they searched the violations of Bell's inequality over limited Hilbert space.

In this paper, we study the nonlocality of multi-qubit GHZ states under three decoherences. The paper is organized as follows. Section II.A introduces our nonlocality criteria, Mermin-Klyshko(MK) inequality, section II.B describes three decoherence models and decohered GHZ states, and section II.C construct the inequalities of the states. In section III, we numerically test the inequalities and discuss the results. Finally, section IV summarizes the conclusions.

II. METHODS

We assume the following situation. Parties are sharing multi-qubit GHZ states, where individual qubit suffers equal and independent local decoherence. To estimate nonlocality, parties measure their qubits in one of two promised measurements and consult the results. Fig. 1 shows the case where the number of party is three. We will use the term "a GHZ state" not only for a three qubit state but also for a multi-qubit GHZ state.

A. How to know nonlocal or not?

We select Mermin-Klyshko(MK)[7] inequalities as nonlocality tester. We want to test the nonlocality of symmetric N -qubit states by dichotomic measurements(two different measurements per party). In that situations, the MK inequality has many advantages; it is simpler than other Bell's inequalities for multi-qubit systems[8],

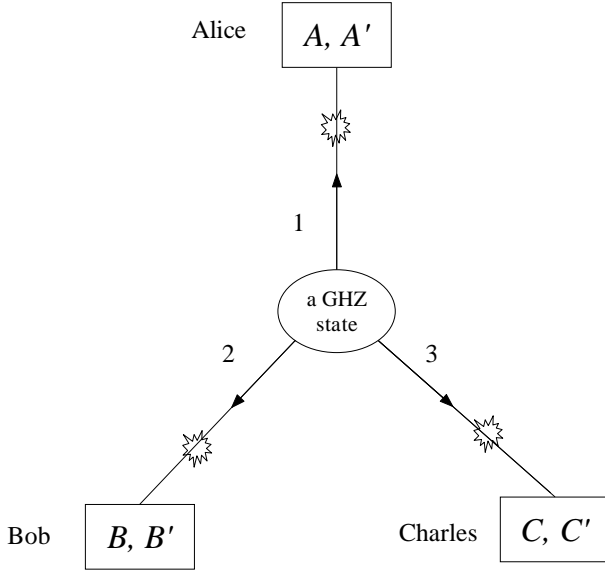


FIG. 1: Experimental Scheme. The source emits a GHZ state $\frac{1}{2}(|000\rangle + |111\rangle)$, where equal and independent local decoherence happens to each individual qubit. The decoherence is represented by flash symbols in the figure. To estimate nonlocality, Alice performs a projection measurement A or A' on her qubit 1, Bob performs a measurement B or B' , and Charles performs a measurement C or C' , respectively.

invariant under permutation, and maximally violated by a GHZ state. However, the inequality also has a limitation; for some nonlocal states, it cannot determine the nonlocality.

MK inequality of an N -qubit system is written as

$$|\langle \mathcal{B}_N \rangle_\psi| \leq 1, \quad (1)$$

where, the operator \mathcal{B}_N is an N -qubit *Bell operator*, and a *Bell value* $\langle \mathcal{B}_N \rangle_\psi$ is the expectation value of it, given a state ψ . Although the bound is 1 as seen in Eq. (1), the Bell values of MK inequalities can rise to $2^{(N-1)/2}$, quantum mechanically. The Bell operator is recursively defined as follows.

$$\mathcal{B}_N \equiv \frac{1}{2}\mathcal{B}_{N-1} \otimes (K + K') + \frac{1}{2}\mathcal{B}'_{N-1} \otimes (K - K'), \quad (2)$$

where the symbol \otimes is vector product. The capital letters stand for individual party's measurements(observables), and i th party's two measurements are presented by the primed and non-primed i th capital ($A, A', B, B', \dots, K, K'$). The letter A (also the lower-case a) refers to first party and the letter K (also k) refers to N th party. The primed Bell operator \mathcal{B}'_N is obtained from \mathcal{B}_N by exchanging every primed and non-primed capitals. The starting of recursion is \mathcal{B}_2 , which is a CHSH inequality, Eq. (3) below. For example, the Bell operators of two, three, and four MK inequality are the following.

$$\mathcal{B}_2 = \frac{1}{2}(AB + AB' + A'B - A'B') \quad (3)$$

$$\mathcal{B}_3 = \frac{1}{2}(ABC' + AB'C + A'BC - A'B'C') \quad (4)$$

$$\begin{aligned} \mathcal{B}_4 = \frac{1}{4}(&-ABCD + ABCD' + ABC'D + ABC'D' \\ &+ AB'CD + AB'CD' + AB'C'D - AB'C'D' \\ &+ A'BCD + A'BCD' + A'BC'D - A'BC'D' \\ &+ A'B'CD - A'B'CD' - A'B'C'D - A'B'C'D') \end{aligned} \quad (5)$$

B. Decoherence Models

Depolarization, dephasing, and dissipation on a qubit are our decoherence models. Let's define p as *the degree of decoherence of an individual qubit*, which lies between 0 and 1, where the value 0 means no decoherence, and 1 means complete decoherence. The depolarization process to a state with a decoherence degree p is represented by

$$|i\rangle\langle j| \rightarrow (1-p)|i\rangle\langle j| + p\delta_{ij}\frac{I}{2}. \quad (6)$$

The dephasing process is described by

$$|i\rangle\langle j| \rightarrow (1-p)|i\rangle\langle j| + p\delta_{ij}|i\rangle\langle j|. \quad (7)$$

The dissipation is the process losing energy, and thus it changes a state to a specific state (e.g., a ground state). We choose $|0\rangle$ as the ground state. The dissipation process is then described by

$$\begin{aligned} |i\rangle\langle i| &\rightarrow (1-p)|i\rangle\langle i| + p|0\rangle\langle 0| \\ |i\rangle\langle j| &\rightarrow (1-p)^{1/2}|i\rangle\langle j| \quad \text{where } i \neq j. \end{aligned} \quad (8)$$

These three decoherences can be mathematically simplified to two processes, population transfer and dephasing.

An initial GHZ state suffers one of the above decoherence processes, which act equally and independently on each qubits. If every individual qubit of a GHZ state is partially depolarized as Eq. (6), then the density matrix of the depolarized GHZ state becomes

$$\begin{aligned} \frac{1}{2}[(1-\frac{p}{2})|0\rangle\langle 0| + \frac{p}{2}|1\rangle\langle 1|]^{\otimes N} &+ (\frac{p}{2}|0\rangle\langle 0| + \\ (1-\frac{p}{2})|1\rangle\langle 1|)^{\otimes N} &+ (1-p)^N(|0\rangle\langle 1|^{\otimes N} + |1\rangle\langle 0|^{\otimes N})], \end{aligned} \quad (9)$$

where $|i\rangle\langle j|^{\otimes N} \equiv |ii\dots i\rangle\langle jj\dots j|$. The partially dephased GHZ state by an amount p is

$$\frac{1}{2}[|0\rangle\langle 0|^{\otimes N} + |1\rangle\langle 1|^{\otimes N} + (1-p)^N(|0\rangle\langle 1|^{\otimes N} + |1\rangle\langle 0|^{\otimes N})], \quad (10)$$

and the partially dissipated GHZ state is

$$\begin{aligned} \frac{1}{2}[|0\rangle\langle 0|^{\otimes N} &+ (p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|)^{\otimes N} \\ &+ (1-p)^{N/2}(|0\rangle\langle 1|^{\otimes N} + |1\rangle\langle 0|^{\otimes N})]. \end{aligned} \quad (11)$$

C. Correlations and MK inequalities

Every projection measurement over a qubit can be represented by a unit vector in the Bloch sphere. An arbitrary observable A can be written as

$$A = (\sigma_x \cos \phi_a + \sigma_y \sin \phi_a) \sin \theta_a + \sigma_z \cos \theta_a, \quad (12)$$

The measurements conducted by the participating parties construct *correlation*, which is the ensemble average of the multiplication of every party's observable $\langle AB \cdots K \rangle \equiv \text{Tr}\{AB \cdots K \rho\}$. The correlation of a GHZ state is

$$\langle AB \cdots K \rangle_{GHZ} = \left[\frac{1 + (-1)^N}{2} \cos \theta_a \cdots \cos \theta_k + \cos(\phi_a + \cdots + \phi_k) \sin \theta_a \cdots \sin \theta_k \right], \quad (13)$$

the correlation of a depolarized GHZ state (9) is

$$\langle AB \cdots K \rangle_{depol} = (1 - p)^N \left[\frac{1 + (-1)^N}{2} \cos \theta_a \cdots \cos \theta_k + \cos(\phi_a + \cdots + \phi_k) \sin \theta_a \cdots \sin \theta_k \right], \quad (14)$$

the correlation of a dephased GHZ state (10) is

$$\langle AB \cdots K \rangle_{dephase} = \left[\frac{1 + (-1)^N}{2} \cos \theta_a \cdots \cos \theta_k + (1 - p)^N \cos(\phi_a + \cdots + \phi_k) \sin \theta_a \cdots \sin \theta_k \right], \quad (15)$$

and the correlation of a dissipated GHZ state (11) is

$$\langle AB \cdots K \rangle_{dissip} = \left[\frac{1 + (2p - 1)^N}{2} \cos \theta_a \cdots \cos \theta_k + (1 - p)^{N/2} \cos(\phi_a + \cdots + \phi_k) \sin \theta_a \cdots \sin \theta_k \right]. \quad (16)$$

Since MK inequalities are sum of correlations, as seen in (3)-(5), the inequalities of a pure GHZ state and those of the state under depolarization (under dephasing or dissipation) can be constructed from (13) and (14) (Eq. (15) or Eq. (16)), respectively.

Comparing the correlations Eq. (13), (14) and (15), we observe that

$$\begin{aligned} \langle AB \cdots K \rangle_{depol} &= (1 - p)^N \langle AB \cdots K \rangle_{GHZ} \\ \langle AB \cdots K \rangle_{dephase|odd} &= (1 - p)^N \langle AB \cdots K \rangle_{GHZ}, \end{aligned} \quad (17)$$

where the last equality holds when the number of qubits is odd. Therefore, Bell inequalities made from the above correlations satisfy

$$\langle \mathcal{B}_N \rangle_{depol} = \langle \mathcal{B}_N \rangle_{dephase|odd} = (1 - p)^N \langle \mathcal{B}_N \rangle_{GHZ}. \quad (18)$$

III. RESULTS AND DISCUSSION

We can easily observe the robustness of multi-qubit nonlocality of a GHZ state under depolarization and the state of odd N under dephasing. The robustness is judged by the amount of the maximum noise for which the Bell inequality is still violated, i.e. nonlocality pro-

σ_x , σ_y , and σ_z are pauli spin matrices, and θ_a and ϕ_a are the polar and azimuthal angles corresponding to A in the Bloch sphere representation. An eigenvalue of the observable (i.e. a measurement result) is 1 or -1.

longs. From Eq. (18) the violation condition of the states is

$$(1 - p)^N \langle \mathcal{B}_N \rangle_{GHZ} > 1. \quad (19)$$

In the above inequality, the noise p becomes maximal at the highest Bell value of a GHZ state $\langle \mathcal{B}_N \rangle_{GHZ}$, which is $2^{(N-1)/2}$. Consequently, the maximum noise p_{max} is given by

$$p_{max} = 1 - 2^{(\frac{1}{N}-1)/2}. \quad (20)$$

Eq. (20) is a monotonically increasing function of N that asymptotically becomes close to $1 - \frac{1}{\sqrt{2}}$, and is shown by the solid line of Fig. 2. In a GHZ state under depolarization and the state of odd N under dephasing, the nonlocality becomes more robust as the number of qubits increases.

We numerically investigated violations of MK inequalities for nontrivial cases: a GHZ state under dissipation and the state of even N under dephasing. We used MATHEMATICA as a simulation tool, and evaluated the system up to five qubits.

The nonlocality of a GHZ state also shows persistency against dissipation with the increasing number of qubits.

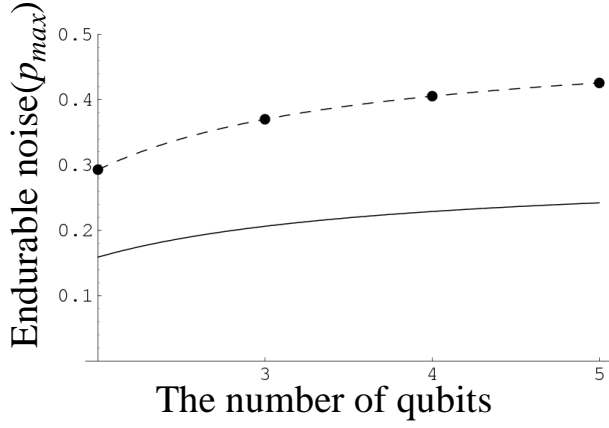


FIG. 2: The noise for which MK inequality is still violated. The solid line is sustainable noise for a GHZ state under depolarization or the state of odd N under dephasing, Eq. (20). The points are for a GHZ state under dissipation by numerical calculations, and fits well to Eq. (21), which is represented by the dashed line.

The maximum noise p_{max} for which the Bell inequality is still violated is numerically calculated and is plotted by the points in Fig. 2. The numerical result fits well to the equation

$$p_{max} = 1 - 2^{\frac{1}{N}-1}. \quad (21)$$

Eq. (21) is also a monotonically increasing function of N and is drawn by the dashed line of Fig. 2.

To see how nonlocality changes, we plot the maximum Bell value of a GHZ state under dissipation in Fig. 3. The curve falls below the inequality bound by dissipation, and it return to the bound at complete dissipation, because the fully dissipated state is a ground state $|00 \cdots 0\rangle$, which has a classical correlation.

Fig. 4 shows that the nonlocality of a GHZ state with even number of qubits is extremely robust against dephasing. Dephasing is considered to be the main cause of destroying quantum coherence. Although dephasing is very fast in a real system, nonlocality always survives, if the dephasing is not complete. This effect was already noted by Cabello and Feito [6] but not for more than two qubits. The extreme robustness of a GHZ state under dephasing critically depends on the parity of the number of qubits. Whether this parity dependence is real or accidental is not clear yet. To study this question further we could explore other nonlocality criteria. We remark that this feature is very closely related to entanglement purification. One can get a pure GHZ state by consuming many partially dephased states by the purification, if the dephasing does not happen completely [9]; this is not the case for depolarization.

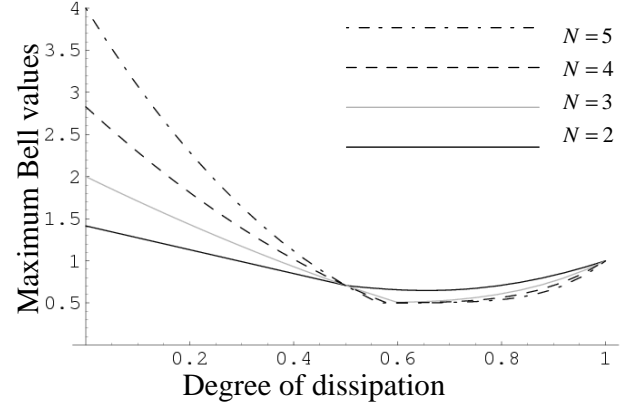


FIG. 3: Maximum Bell value of the MK inequality as a function of p for states with dissipation given by eq. (8). The number of qubits is from two to five and is distinguished by line style.

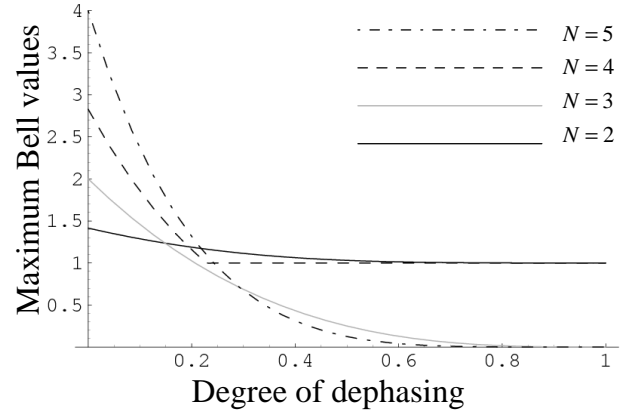


FIG. 4: Maximum Bell value of the MK inequality as a function of p for states under dephasing given by eq. (7). The number of qubits is from two to five and is distinguished by line style.

IV. CONCLUSION

In summary, we studied the persistency of nonlocality over N -qubit GHZ states under realistic local decoherences: depolarization, dephasing, and dissipation. We observed nonlocality by testing violations of MK inequality of decohered GHZ states, using numerical simulations. This study revealed that multi-qubit GHZ states have much robust nonlocality with increasing number of qubits, and showed extreme persistency of a GHZ state of even N against dephasing. This strong persistency can not be seen for a GHZ state of odd N . Whether this parity dependence is real nature of nonlocality or due to the incompleteness of nonlocality criterion needs further investigation. The states, having robust nonlocality, will be very useful in quantum communication and quantum information with realistic noisy circumstances.

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- [1] J. S. Bell, *Physics* **1**, 195 (1964).
 - [2] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, *Phys.Rev.Lett.* **23**, 880 (1969).
 - [3] R. F. Werner, *Phys.Rev.A* **40**, 4277 (1989).
 - [4] D. Kazlikowski, P. Ginacinski, M. Zukowski, W. Miklaszewski, and A. Zeilinger, *Phys.Rev.Lett.* **85**, 4418 (2000).
 - [5] C. Simon and J. Kempe, *Phys.Rev.A* **65**, 052327 (2002).
 - [6] A. Cabello and A. Feito, *Phys.Rev.A* **72**, 052112 (2005).
 - [7] N. D. Mermin, *Phys.Rev.Lett.* **65**, 1838 (1990);
M. Ardehali, *Phys.Rev.A* **46**, 5375 (1992);
A. V. Belinskii and D. N. Klyshko, *Phys.Usp.* **36**, 653 (1993).
 - [8] M. Zukowski and C. Brukner, *Phys.Rev.Lett.* **88**, 210401 (2002);
R. F. Werner and M. W. Wolf, *Phys.Rev.A* **64**, 032112 (2001);
W. Laskowski, T. Paterek, M. Zukowski, and C. Brukner, *Phys.Rev.Lett.* **93**, 200401 (2004);
T. Paterek, W. Laskowski, and M. Zukowski, *Mod.Phys.Lett.A* **21**, 111 (2006).
 - [9] M. Murao, M. B. Plenio, S. Popescu, V. Vedral, and P. L. Knight, *Phys.Rev.A* **57**, R4075 (1998).